

APPLICATIONS OF RADON-NIKODÝM THEOREMS TO MARTINGALE CONVERGENCE

BY

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Introduction. The subject of convergence of martingales of functions with values in a Banach space was first treated by Scalora [17] and Chatterji [3] who independently showed that a martingale of functions with values in a reflexive Banach space obeys roughly the same basic convergence theorems as martingales of real or complex valued functions. Various authors have continued the work; among them being A. and C. Ionescu-Tulcea [9], who generalized the earlier work by considering martingale convergence theorems from the point of view of abstract ergodic theory; Chatterji [4], who generalized his earlier work by making certain assumptions on the range space of the martingale of functions; and Métivier [14] and [15] who has treated in detail not only the earlier problems but also related problems dealing with both weak and strong martingales and martingales with values in locally convex linear topological spaces.

In one of his papers [15, Theorem 11] Métivier proved a theorem characterizing those Banach space valued measures which admit a representation as an indefinite Bochner integral with respect to a finite nonnegative real valued measure. More recently, Rieffel [16] gave a different, but related, characterization of such vector valued measures. Since the connection between the theory of derivations and the theory of convergence of martingales is well established, a natural problem is to use the above mentioned Radon-Nikodým theorems of Métivier and Rieffel to obtain a characterization of norm convergent Banach valued martingales and other results. The purpose of this paper is to do exactly this.

The first section is concerned with preliminaries which establish the setting for the work which follows. The second section, which is the main section of the paper, is devoted to characterization of norm convergent martingales in certain Orlicz spaces L^Φ of vector valued functions. The third section continues with an extension of theorems of Doob, Krickeberg and Chatterji dealing with measure convergence and almost everywhere convergence of $L^1(\mathfrak{X})$ -bounded martingales. Finally, in §4, the Radon-Nikodým-Bochner theorem for finitely additive measures is generalized to the vector valued case.

1. **Some preliminaries.** Throughout this paper, unless otherwise noted, (Ω, Σ, μ) is a fixed finite measure space; i.e. μ is a countably additive nonnegative real valued

finite set function defined on a σ -field Σ of subsets of the point set Ω . \mathfrak{X} is a Banach space, Φ is a Young's function [22] with complementary function Ψ . $L^\Phi(\Omega, \Sigma, \mu, \mathfrak{X})$ ($=L^\Phi(\mu, \mathfrak{X})$) is the collection of all totally μ -measurable [6, III.2.10] \mathfrak{X} -valued functions f which satisfy $\int_\Omega \Phi(\|f\|/k) d\mu < \infty$ for some $k > 0$. Upon the introduction of the equivalent norms

$$N_\Phi(f) = \inf \left\{ k > 0 : \int_\Omega \Phi(\|f\|/k) d\mu \leq 1 \right\}$$

or

$$\|f\|_\Phi = \sup \left\{ \int_\Omega \|f\| g d\mu : g > 0, \int_\Omega \Psi(g) d\mu \leq 1 \right\}$$

[10, pp. 19, 22] and the identification of functions which differ on at most a μ -null set, $L^\Phi(\mu, \mathfrak{X})$ becomes a Banach space. If there exists a constant k such that $\Phi(2x) \leq k\Phi(x)$ for all real x , Φ is said to obey the Δ_2 -condition. $L^1 \supset L^\Phi$, and all L^p spaces $1 \leq p < \infty$ are found among the L^Φ spaces with Φ obeying the Δ_2 -condition. When Φ obeys the Δ_2 -condition, simple functions are dense in $L^\Phi(\mu, \mathfrak{X})$. In addition

LEMMA 1. *Let \mathcal{F} be a subfield of Σ such that the σ -field generated by \mathcal{F} is Σ . If Φ obeys the Δ_2 -condition, then the linear span of the set*

$$\{x\chi_E : x \in \mathfrak{X}, E \in \mathcal{F}\}$$

is dense in $L^\Phi(\mu, \mathfrak{X})$. (Here $\chi_E(\omega) = +1$, $\omega \in E$; $\chi_E(\omega) = 0$, $\omega \notin E$.)

The proof, which will be omitted, follows quickly from the facts that simple functions are dense in $L^\Phi(\mu, \mathfrak{X})$, and that $\lim_{\mu(E) \rightarrow 0} N_\Phi(x\chi_E) = 0$ for all $x \in \mathfrak{X}$, and that for $E \in \Sigma$, there exists [6, III.71] $\{E_n\} \subset \mathcal{F}$ such that $\lim_{n \rightarrow \infty} \mu(E_n \triangle E) = 0$ where $E_n \triangle E = (E_n - E) \cup (E - E_n)$.

DEFINITION 1. Let $f \in L^1(\mu, \mathfrak{X})$ and B be a sub- σ -field of Σ . A function $g \in L^1(\mu, \mathfrak{X})$ is the conditional expectation of f given B , if g is B -measurable and $\int_E f d\mu = \int_E g d\mu$ for all $E \in B$. In this case one writes $E^B(f) = g$. Here, as throughout, the integral is that of [6, Chapter III].

The operator E^B is defined for all $f \in L^1(\mu, \mathfrak{X})$ and when considered as an operator $E^B: L^\Phi(\mu, \mathfrak{X}) \rightarrow L^\Phi(\mu, \mathfrak{X})$ is a linear contraction.

DEFINITION 2. Let T be a directed set and $\{B_\tau, \tau \in T\}$ be an increasing net of sub- σ -fields of Σ ; i.e. $\tau_1 \leq \tau_2$ implies $B_{\tau_1} \subset B_{\tau_2}$. $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale in $L^\Phi(\mu, \mathfrak{X})$ if each $f_\tau \in L^\Phi(\mu, \mathfrak{X})$ and $\tau_1 \leq \tau_2$ implies $E^{B_{\tau_1}}(f_{\tau_2}) = f_{\tau_1}$.

This section will be terminated with a lemma which will be employed crucially in the next section. It is only a slight modification of a well known result.

LEMMA 2. *Let Φ obey the Δ_2 -condition. A martingale $\{f_\tau, B_\tau, \tau \in T\}$ converges in the L^Φ norm if and only if there exists a function $f \in L^\Phi(\mu, \mathfrak{X})$ such that*

$$\lim_\tau \int_E f_\tau d\mu = \int_E f d\mu$$

for all $E \in \bigcup_\tau B_\tau$.

Proof (Sufficiency). Since $\{B_\tau, \tau \in T\}$ is an increasing net of σ -fields, it follows easily that $\bigcup_\tau B_\tau$ is a subfield of Σ . By standard techniques, without loss of generality it will be assumed that the σ -field generated by $\bigcup_\tau B_\tau$ is Σ . Now let $\varepsilon > 0$ be given. By Lemma 1, there exists a function $f_\varepsilon = \sum_{i=1}^n x_i \chi_{E_i} : x_i \in \mathfrak{X}, E_i \in \bigcup_\tau B_\tau$, such that $N_\Phi(f - f_\varepsilon) < \varepsilon/2$. Since $\{B_\tau, \tau \in T\}$ is an increasing net, there exists a $\tau_0 \in T$ such that for all $\tau \geq \tau_0$ $\{E_i\}_{i=1}^n \subset B_\tau$. Hence for $\tau \geq \tau_0$ $E^{B_\tau}(f_\varepsilon) = f_\varepsilon$. Moreover for $\tau \geq \tau_1$, $\int_E f_\tau d\mu = \int_E f_{\tau_1} d\mu$ for $E \in B_{\tau_1}$ by the martingale property. It follows that $E^{B_\tau}(f) = f_\tau$.

Therefore for $\tau \geq \tau_0$,

$$N_\Phi(f - f_\tau) \leq N_\Phi(f - f_\varepsilon) + N_\Phi(f_\varepsilon - f_\tau) = N_\Phi(f - f_\varepsilon) + N_\Phi(E^{B_\tau}(f_\varepsilon - f)) \leq 2N_\Phi(f - f_\varepsilon),$$

since E^{B_τ} is a contraction, $< 2\varepsilon/2 = \varepsilon$, by the choice of f_ε . This establishes the sufficiency of the condition.

To prove the necessity, suppose $\lim_\tau f_\tau = f$ in N_Φ norm. Then since the operation defined for $g \in L^\Phi(\mu, \mathfrak{X})$ by $g \rightarrow \int_E g d\mu$, is a bounded linear operator for each $E \in \Sigma$, it follows that

$$\lim_\tau \int_E f_\tau d\mu = \int_E f d\mu$$

for all $E \in \bigcup_\tau B_\tau$. This proves the necessity and the lemma.

2. The mean convergence of Banach valued martingales. This section is devoted to characterizing those martingales $\{f_\tau, B_\tau, \tau \in T\} \subset L^\Phi(\mu, \mathfrak{X})$ such that

$$\lim_\tau N_\Phi(f_\tau - f) = 0$$

for some $f \in L^\Phi(\mu, \mathfrak{X})$ when Φ obeys the Δ_2 -condition. Some results along these lines have been obtained by Chatterji [3] and [4], Scalora [17], A. and C. Ionescu-Tulcea [9], and Métivier [14] and [15]. Although the conditions given in Theorem 2 below bear some relation to those of Métivier [15, Theorem 4, p. 190] it is believed that Theorem 2 is the first theorem which provides a necessary and sufficient condition for a martingale to converge in the mean. The proof of that theorem hinges on the following proposition which is basic to all of the considerations of this paper. Essentially, it translates the Radon-Nikodým theorems of Métivier [15, Theorem 11] and Rieffel [16] into a form usable in the present context.

(In the statement of Proposition 1 below, the wording "...weakly (norm) compact..." refers to two *separate* conditions. The proposition is true if read "...a weakly compact..." and it is equally true if read "...a norm compact...". Thus Proposition 1 is actually two separate propositions. This wording is employed in the interest of economy of space and will be used in other statements in this paper.)

PROPOSITION 1. *Let Σ_0 be subfield of Σ such that Σ is the σ -field generated by Σ_0 .*

Let μ be a finite nonnegative measure on Σ . If F is a countably additive \mathfrak{X} -valued measure defined on Σ_0 , then F admits the representation

$$F(E) = \int_E f d\mu, \quad E \in \Sigma_0$$

for some $f \in L^1(\mu, \mathfrak{X})$ if and only if

- (i) F is of bounded variation;
- (ii) F is μ -continuous; i.e.

$$\lim_{\mu(E) \rightarrow 0} \|F(E)\| = 0,$$

and

(iii) Given $\varepsilon > 0$ there exists a weakly (norm) compact convex subset $K \subset \mathfrak{X}$ such that for any $\delta > 0$ there exists $E_0 (= E_0(\varepsilon, K, \delta)) \in \Sigma_0$ with $\mu(\Omega - E_0) < \varepsilon$ and $F(E) \in \mu(E)K + \delta U$ for all $E \subset E_0$, $E \in \Sigma_0$, where U is the closed unit ball $\{\|x\| \leq 1\}$ of \mathfrak{X} .

Proof (Sufficiency). Let F satisfy (i), (ii), and (iii). The argument of [19, p. 39] shows that since F is μ -continuous, F has a μ -continuous (hence countably additive) extension (still denoted by F) of bounded variation to Σ . Now let $\varepsilon > 0$ be given and by (iii) choose a convex weakly (norm) compact set K and sets $\{E_n\} \subset \Sigma_0$, $\mu(\Omega - E_n) < \varepsilon$ such that

$$F(E) \in \mu(E)K + n^{-2}U$$

for $E \subset E_n$, $E \in \Sigma_0$, $n = 1, 2, \dots$. Now consider $E_0 = \limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$. By standard results of measure theory [8]

$$\mu(\Omega) - \varepsilon \leq \limsup_n \mu(E_n) \leq \mu(E_0).$$

That is, $\mu(\Omega - E_0) < \varepsilon$. Also if $E \subset E_0$, $E \in \Sigma$, then $E \subset \bigcup_{m=n}^{\infty} E_m$ for each n . Hence if n is fixed and $S_n = E_n$, $S_{n+1} = E_{n+1} - E_n, \dots, S_{n+k+1} = E_{n+k+1} - (\bigcup_{m=n}^{n+k} E_m), \dots$ then the sequence $\{S_m\}_{m=n}^{\infty}$ is disjoint and $E \subset \bigcup_{m=n}^{\infty} S_m$. Moreover since F on Σ is countably additive,

$$F(E) = F\left(E \cap \left(\bigcup_{m=n}^{\infty} S_m\right)\right) = \sum_{m=n}^{\infty} F(E \cap S_m).$$

Next note that $E \cap S_m \subset S_m \subset E_m$ for all $m \geq n$. Therefore, according to [6, III. 7.1], there exists for each m , a sequence $\{H_{mj}\} \subset \Sigma_0$, which may be assumed to satisfy $\{H_{mj}\} \subset E_m$, such that $\lim_j \mu((E \cap S_m) \triangle H_{mj}) = 0$. The μ -continuity of F and the identities

$$\mu(G_1 \triangle G_2) = \mu(G_1 - G_2) + \mu(G_2 - G_1)$$

and

$$F(G_1) - F(G_2) = F(G_1 - G_2) - F(G_2 - G_1)$$

guarantee that $\lim_j F(H_{mj}) = F(E \cap S_m)$ strongly in \mathfrak{X} , and $\lim_j \mu(H_{mj}) = \mu(E \cap S_m)$, $m \geq n$. Since $H_{mj} \subset E_m$, $x_{mj} \in K$ and $y_{mj} \in U$ can be chosen such that

$$F(H_{mj}) = \mu(H_{mj})x_{mj} + \frac{y_{mj}}{m^2}, \quad m \geq n, j \geq 1.$$

Since K is weakly (norm) compact, the Eberlein-Šmulian theorem guarantees that it can (and will) be assumed without loss of generality that $\lim_j x_{nj} = x_n \in K$ exists in the weak topology of \mathfrak{X} . But then for $m \geq n$

$$\begin{aligned} \|F(E \cap S_m) - \mu(E \cap S_m)x_m\| &\leq \liminf_j \|F(H_{mj}) - \mu(H_{mj})x_{mj}\| \\ &\leq \limsup_j \|m^{-2}y_{mj}\| \leq m^{-2}. \end{aligned}$$

Hence for each $m \geq n$, one can write

$$F(E \cap S_m) = \mu(E \cap S_m)W_m + m^{-2}Z_m$$

where $W_m \in K$ and $Z_m \in U$. Then

$$\begin{aligned} F(E) &= \sum_{m=n}^{\infty} F(E \cap S_m) = \sum_{m=n}^{\infty} (\mu(E \cap S_m)W_m + m^{-2}Z_m) \\ &= \sum_{m=n}^{\infty} \mu(E \cap S_m)W_m + \sum_{m=n}^{\infty} m^{-2}Z_m, \end{aligned}$$

since both series are absolutely convergent. Rearranging yields

$$\left\| F(E) - \sum_{m=n}^{\infty} \mu(E \cap S_m)W_m \right\| = \left\| \sum_{m=n}^{\infty} m^{-2}Z_m \right\| \leq \sum_{m=n}^{\infty} m^{-2}.$$

Now, note that

$$\sum_{m=n}^{\infty} \mu(E \cap S_m)W_m = \mu(E) \sum_{m=n}^{\infty} \frac{\mu(E \cap S_m)}{\mu(E)} W_m \quad (0/0 = 0).$$

But since K is convex and closed, and $\sum_{m=n}^{\infty} \mu(E \cap S_m)/\mu(E) = 1$,

$$K_n = \sum_{m=n}^{\infty} \frac{\mu(E \cap S_m)}{\mu(E)} W_m \in K.$$

Moreover,

$$0 \leq \lim_{n \rightarrow \infty} \|F(E) - \mu(E)K_n\| \leq \lim_{n \rightarrow \infty} \sum_{m=n}^{\infty} m^{-2} = 0.$$

It follows that $F(E) \in \mu(E)K$ for all $E \subset E_0$, $E \in \Sigma$. This shows that in the case K is weakly compact that F satisfies the hypothesis of the Radon-Nikodým theorem of Métivier [15, Theorem 11, p. 203], and in the case K is norm compact F satisfies the Radon-Nikodým theorem of Rieffel [16, p. 466]. In any case there exists $f \in L^1(\mu, \mathfrak{X})$ such that

$$F(E) = \int_E f d\mu, \quad E \in \Sigma$$

and in particular for $E \in \Sigma_0$. This completes the proof of the sufficiency.

(Necessity). Suppose there exists $f \in L^1(\mu, \mathfrak{X})$ such that

$$F(E) = \int_E f d\mu, \quad E \in \Sigma_0.$$

Then F has an obvious "natural" extension to Σ , still denoted by F . Standard properties of the integral show that F is μ -continuous. Furthermore, it is well known that the variation of F is just $\int_{\Omega} \|f\| d\mu < \infty$. Hence (i) and (ii) are satisfied. Now let $\varepsilon > 0$ be given. According to the Radon-Nikodým theorem of Rieffel [16, Proposition 1.12, p. 472] there exists a set $E_0 \in \Sigma$, $\mu(\Omega - E_0) < \varepsilon/2$ and a norm compact (hence weakly compact) set K , which by Mazur's theorem may be assumed convex, such that $F(E) \in \mu(E)K$, $E \subset E_0$, $E \in \Sigma$. Next, let $\delta > 0$ be given and by [6, III.7.1], choose $E_\delta \in \Sigma_0$ subject to $\mu(E_0 \triangle E_\delta) < \beta$ where $\beta = \min \{\varepsilon/2, \delta/2m, \alpha\}$ and $m-1 = \max \{\|x\| : x \in K\}$, and $\mu(E) < \infty$, $\alpha > 0$, implies $\|F(E)\| < \delta/2$. Then $\mu(\Omega - E) \leq \mu(\Omega - E_0) + \mu(E_0 \triangle E_\delta) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Now let $E \subset E_\delta$, $E \in \Sigma_0$. By hypothesis there exists $x \in K$ such that $F(E \cap E_0) = \mu(E \cap E_0)x$. For this x , one has

$$\begin{aligned} \|F(E) - \mu(E)x\| &\leq \|F(E) - F(E \cap E_0)\| + \|\mu(E \cap E_0)x - \mu(E)x\| \\ &\leq \|F(E - E_0)\| + \|x\|\mu(E - E_0) < \delta/2 + \delta/2 \end{aligned}$$

by the choice of β and the fact that $E - E_0 \subset E_\delta - E_0$. This proves the necessity of the condition and the theorem.

The following result, which characterizes mean convergent martingales in certain $L^\Phi(\mu, \mathfrak{X})$ spaces is the main result of this paper.

THEOREM 2. *Let Φ obey the Δ_2 -condition and let $\{f_\tau, B_\tau, \tau \in T\}$ be a martingale in $L^\Phi(\mu, \mathfrak{X})$. The net $\{f_\tau, \tau \in T\}$ is convergent in the $L^\Phi(\mu, \mathfrak{X})$ norm if and only if*

- (i) *there exists a constant $m < \infty$ such that $\sup_{\tau \in T} N_\Phi(f_\tau) \leq m$;*
- (ii) *$\{f_\tau, \tau \in T\}$ is terminally uniformly integrable; i.e. given $\varepsilon > 0$ there is a $\delta > 0$ and an index $\tau_0 \in T$ such that $\mu(E) < \delta$ implies $\int_E \|f_\tau\| d\mu < \varepsilon$, $\tau \geq \tau_0$; and*
- (iii) *given $\varepsilon > 0$ there exists a weakly (norm) compact convex set $K \subset \mathfrak{X}$ such that for any $\delta > 0$ there is an index $\tau_0 \in T$ and a set $E_0 \in B_{\tau_0}$, $\mu(\Omega - E_0) < \varepsilon$ satisfying $\tau \geq \tau_0$ implies*

$$\int_E f_\tau d\mu \in \mu(E)K + \delta U$$

for all $E \subset E_0$, $E \in B_\tau$.

In the case that Ψ , the Young's function complementary to Φ , satisfies $\Psi'(x) < \infty$ for $x < \infty$ (ii) above is implied by (i) and may be dropped.

Proof. Without loss of generality, it will be assumed throughout the entire proof that Σ is generated by $\bigcup_\tau B_\tau$.

(Necessity). Suppose that $\lim_\tau f_\tau = f$ in the L^Φ topology. Then $\lim_\tau N_\Phi(f_\tau) = N_\Phi(f)$. On the other hand since E^B is a contraction, the fact that $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale ensures that $\{N_\Phi(f_\tau), \tau \in T\}$ is an increasing net. Therefore $N_\Phi(f_\tau) \leq$

$N_0(f)$. This proves (i). To prove (ii), note that L^Φ convergence implies L^1 convergence. Hence

$$\lim_{\tau} \left| \int_E \|f_{\tau}\| d\mu - \int_E \|f\| d\mu \right| \leq \lim_{\tau} \int_E \|f_{\tau} - f\| d\mu \leq \lim_{\tau} \|f_{\tau} - f\|_1 = 0$$

uniformly in $E \in \Sigma$. (ii) follows immediately from the μ -continuity of the indefinite integral $\int_{(\cdot)} \|f\| d\mu$. To prove (iii), let $\varepsilon > 0$ be given. By Proposition 1, there is a weakly (norm) compact convex subset $K \subset \mathfrak{X}$ such that given any $\delta > 0$ there is an $E_{\delta} \in \bigcup_{\tau} B_{\tau}$ such that

$$\int_E f d\mu \in \mu(E)K + \delta U,$$

$E \subset E_{\delta}$, $E \in \bigcup_{\tau} B_{\tau}$. For a fixed $\delta > 0$ select $\tau_0 \in T$ such that $E_{\delta} \in B_{\tau_0}$. Then for $\tau_0 \leq \tau$ and $E \in B_{\tau}$, $E \subset E_{\delta}$, one has, by the martingale property

$$\int_E f_{\tau} d\mu = \int_E f d\mu \in \mu(E)K + \delta U.$$

This proves the necessity.

(Sufficiency). The sufficiency will be proven first for $L^1(\mu, \mathfrak{X})$. Define the set function F on $\bigcup_{\tau} B_{\tau}$ by

$$F(E) = \lim_{\tau} \int_E f_{\tau} d\mu, \quad E \in \bigcup_{\tau} B_{\tau}.$$

The martingale property ensures that the defining net on the right is eventually constant for $E \in \bigcup_{\tau} B_{\tau}$, and hence F is well defined and obviously finitely additive. From (i), for any finite disjoint collection $\{E_n\}_{n=1}^m \subset \bigcup_{\tau} B_{\tau}$,

$$\begin{aligned} \sum_{n=1}^m \|F(E_n)\| &= \sum_{n=1}^m \left\| \lim_{\tau} \int_{E_n} f_{\tau} d\mu \right\| \\ &= \sum_{n=1}^m \left\| \int_{E_n} f_{\tau_0} d\mu \right\| \leq \int_{\Omega} \|f_{\tau_0}\| d\mu \leq M \end{aligned}$$

for $\tau_0 \in T$ chosen such that $\{E_n\}_{n=1}^m \subset B_{\tau_0}$. Hence F is of bounded variation. (ii) clearly implies the μ -continuity, and therefore the countable additivity of F .

To complete the proof of the sufficiency in the $L^1(\mu, \mathfrak{X})$ case, by Lemma 2, it need only be shown that there exists $f \in L^1(\mu, \mathfrak{X})$ such that

$$F(E) = \lim_{\tau} \int_E f_{\tau} d\mu = \int_E f d\mu, \quad E \in \bigcup_{\tau} B_{\tau}.$$

This is clearly the same as showing F has a Radon-Nikodým derivative. Let $\varepsilon > 0$ be given and let K be as in (iii). By (iii) for any fixed choice of $\delta > 0$ there exists an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$, $\mu(\Omega - E_0) < \varepsilon$ such that $\int_E f_{\tau} d\mu \in \mu(E)K + \delta U$ for $E \subset E_0$,

$E \in B_\tau$, $\tau \geq \tau_0$. So, let $E \subset E_0$, $E \in \bigcup_\tau B_\tau$ be arbitrary, then $E \in B_{\tau_0}$ for some $\tau_1 \geq \tau_0$ and

$$F(E) = \int_E f_{\tau_1} d\mu \in \mu(E)K + \delta U.$$

Hence, by Proposition 1, there exists $f \in L^1(\mu, \mathfrak{X})$ such that

$$F(E) = \int_E f d\mu = \lim_\tau \int_E f_\tau d\mu,$$

for $E \in \bigcup_\tau B_\tau$. This, combined with Lemma 2 completes the proof in the $L^1(\mu, \mathfrak{X})$ case.

Moving to the general case, assume (i), (ii) and (iii) hold. Since $\mu(\Omega) < \infty$, N_Φ dominates the $L^1(\mu, \mathfrak{X})$ norm, and the martingale converges in $L^1(\mu, \mathfrak{X})$, by the above, to a function $f \in L^1(\mu, \mathfrak{X})$. Clearly f satisfies

$$\lim_\tau \int_E f_\tau d\mu = \int_E f d\mu \quad \text{for } E \in \bigcup_\tau B_\tau.$$

Since $\{f_\tau, \tau \in T\}$ converges to f in $L^1(\mu, \mathfrak{X})$ norm, there exists a sequence of indices $\{\tau_n\} \subset T$ such that $\lim_n \|f_{\tau_n} - f\|_1 = 0$ and such that $\lim_n f_{\tau_n}(\omega) = f(\omega)$ a.e. But since $N_\Phi(f_{\tau_n}) \leq M$

$$\int_\Omega \Phi(\|f_{\tau_n}\|/(M+1)) d\mu \leq 1 \quad \text{for all } n,$$

and by Fatou's lemma,

$$\int_\Omega \Phi(\|f\|/(M+1)) d\mu \leq \liminf_n \int_\Omega \Phi(\|f_{\tau_n}\|/(M+1)) d\mu \leq 1.$$

Hence $f \in L^\Phi(\mu, \mathfrak{X})$, and since $\lim_\tau \int_E f_\tau d\mu = \int_E f d\mu$, $E \in \bigcup_\tau B_\tau$, an application of Lemma 2 concludes the proof of the sufficiency.

To complete the proof of the theorem, suppose Ψ , the Young's function complementary to Φ satisfies $\Psi(x) < \infty$ for $x < \infty$. Assuming (i), one has

$$\int_E \|f_\tau\| d\mu = \int_\Omega \|f_\tau\|_{\chi_E} d\mu \leq N_\Phi(f_\tau) \|\chi_E\|_\Psi \leq M \|\chi_E\|_\Psi.$$

But $\lim_{\mu(E) \rightarrow 0} \|\chi_E\|_\Psi = 0$, provided $\Psi(x) < \infty$ for $x < \infty$. Hence the net $\{f_\tau, \tau \in T\}$ is uniformly integrable and *a fortiori* satisfies (ii). Q.E.D.

Specializing Theorem 2 to the case where the index set T is N , the positive integers with the natural order, is the following theorem.

THEOREM 3. *Let $\{f_n, B_n\}$ be a martingale satisfying the hypothesis of Theorem 2. If f_∞ is the L^1 -limit of $\{f_n\}$, then $\lim_{n \rightarrow \infty} f_n(\omega) = f_\infty(\omega)$ for μ -almost all $\omega \in \Omega$.*

Proof. This is a consequence of a theorem [9, p. 189] due to A. and C. Ionescu-Tulcea which asserts that a mean convergent martingale converges almost everywhere to its L^1 -limit. Q.E.D.

Theorem 3 can be weakened somewhat. This is the topic of §3.

3. $L^1(\mu, \mathfrak{X})$ -bounded martingales. This section is concerned with supplying sufficient conditions for an $L^1(\mu, \mathfrak{X})$ -bounded martingale to converge in measure and almost everywhere. The classical result in this direction is due first to Doob and later to Krickeberg who proved that if $\{f_\tau, B_\tau, \tau \in T\}$ is a martingale of real valued functions with $\|f_\tau\|_1 \leq M < \infty$ then the net $\{f_\tau, \tau \in T\}$ converges in measure to a function $f \in L^1(\mu)$ and if $T = N$, the positive integers, then this convergence is also μ -almost everywhere. Recently, in the case $T = N$, Chatterji [4, Theorem 2] generalized this to \mathfrak{X} -valued functions by placing certain convenient assumptions on the Banach space \mathfrak{X} . In this section results are obtained by placing assumptions on the martingale itself. It will be shown that the results here subsume all of the above mentioned results.

THEOREM 4. Let $\{f_\tau, B_\tau, \tau \in T\}$ be a martingale in $L^1(\mu, \mathfrak{X})$ satisfying

(i) $\sup_\tau \int_\Omega \|f_\tau\| d\mu \leq M < \infty$; and

(ii) for each $\varepsilon > 0$ there exists a weakly compact convex set K such that given $\delta > 0$ there is an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$, $\mu(\Omega - E_0) < \varepsilon$ such that for $\tau \geq \tau_0$

$$\int_E f_\tau d\mu \in \mu(E)K + \delta U$$

for all $E \subset E_0$, $E \in B_\tau$, then there exists $f \in L^1(\mu, \mathfrak{X})$ such that $\lim_\tau f_\tau = f$ in μ -measure. If, in addition, $T = N$, then $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$, strongly in \mathfrak{X} , a.e. $[\mu]$ as well.

Proof. As in the proof of Theorem 2, define F on the field $\bigcup_\tau B_\tau$ by

$$F(E) = \lim_\tau \int_E f_\tau d\mu, \quad E \in \bigcup_\tau B_\tau.$$

Then, as in the proof of Theorem 2, F is finitely additive, and (i) implies that F is of bounded variation. Following standard methods [4, p. 57], one can write $F = G + H$, where G and H are \mathfrak{X} -valued finitely additive set functions of bounded variation where $|G|$, the variation of G is singular with respect to μ (i.e. given $\varepsilon > 0$, $\delta > 0$ there exists $E \in \bigcup_\tau B_\tau$, $\mu(E) < \varepsilon$, $|G|(\Omega - E) < \delta$), and H is a countably additive set function which is μ -continuous. First it will be shown that there is an $h \in L^1(\mu, \mathfrak{X})$ such that $H(E) = \int_E h d\mu$, $E \in \bigcup_\tau B_\tau$.

For this, note that H satisfies (i) and (ii) of the hypothesis of Proposition 1. To show that H satisfies (iii) of that hypothesis, let $\varepsilon > 0$ be given and choose a weakly compact convex set K such that for any $\delta > 0$ there is an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$, $\mu(\Omega - E_0) < \varepsilon/2$ such that $\tau \geq \tau_0$ implies

$$\int_E f_\tau d\mu \in \mu(E)K + (\delta/2)U,$$

$E \subset E_0$, $E \in B_\tau$. Now let $\delta > 0$ be fixed. Choose the corresponding index $\tau_0 \in T$ and set $E_0 \in B_{\tau_0}$ as above; also choose $\tilde{E} \in \bigcup_\tau B_\tau$ such that $\mu(\tilde{E}) < \varepsilon/2$ and $|G|(\Omega - \tilde{E}) < \delta/2$. Setting $E_1 = E_0 \cap (\Omega - \tilde{E})$, and choosing $\tau_1 \geq \tau_0$ such that $E_1 \in B_{\tau_1}$ one has $\mu(\Omega - E_1) < \varepsilon$ and for $\tau \geq \tau_1$

$$\int_E f_\tau d\mu \in \mu(E)K + (\delta/2)U,$$

$E \in B_\tau$, $E \subset E_1 \subset E_0$. But for $E \in B_\tau$, $F(E) = \int_E f_\tau d\mu$, by the martingale property. Hence

$$F(E) \in \mu(E)K + (\delta/2)U,$$

$E \subset E_1$. But $\|H(E) - F(E)\| \leq \|G(E)\| \leq |G|(F_1) \leq |G|(\Omega - \tilde{E}) < \delta/2$, $E \subset E_1$, $E \in \bigcup_\tau B_\tau$. Hence

$$H(E) \in \mu(E)K + \delta U.$$

This shows H satisfies (iii) of Proposition 1 and therefore establishes the existence of $h \in L^1(\mu, \mathfrak{X})$ such that

$$H(E) = \int_E h d\mu, \quad E \in \bigcup_\tau B_\tau.$$

Next, define h_τ by $h_\tau = E^{B_\tau}(h)$ and note that for $E \in B_\tau$, $F(E) = \int_E f_\tau d\mu$. Hence for $E \in B_\tau$, $G(E) = \int_E (f_\tau - h_\tau) d\mu$. Since $f_\tau - h_\tau$ is B_τ -measurable there exists a B_τ -measurable $g_\tau (= (f_\tau - h_\tau))$ such that $G(E) = \int_E g_\tau d\mu$, $E \in B_\tau$. Also since $\{f_\tau, B_\tau, \tau \in T\}$ and $\{h_\tau, B_\tau, \tau \in T\}$ are martingales, so is $\{g_\tau, B_\tau, \tau \in T\}$. Moreover, $\{h_\tau, B_\tau, \tau \in T\}$ is an L^1 -convergent martingale by Lemma 2 since $\lim_\tau \int_E h_\tau d\mu = \int_E h d\mu$, $E \in B_\tau$. Since this martingale is convergent in $L^1(\mu, \mathfrak{X})$, it is also convergent in μ -measure to $h_0 \in L^1(\mu, \mathfrak{X})$.

Finally, it will be shown that $\lim_\tau g_\tau = 0$ in μ -measure. For this, let $0 < \varepsilon < 1$ and $\delta > 0$ be given. Since $|G|$ and μ are singular, there exists $E \in \bigcup_\tau B_\tau$ such that

$$\mu(\Omega - E) + |G|(E) < \delta\varepsilon/2.$$

Now choose an index $\tau_0 \in T$ such that $E \in B_{\tau_0}$. Then for any $\tau \geq \tau_0$

$$\begin{aligned} \mu(\{\omega : \|g_\tau(\omega)\| > \varepsilon\}) &= \mu(\{\omega \notin E : \|g_\tau(\omega)\| > \varepsilon\}) + \mu(\{\omega \in E : \|g_\tau(\omega)\| > \varepsilon\}) \\ &\leq \mu(\Omega - E) + (1/\varepsilon) \int_{E \cap \{\omega : \|g_\tau(\omega)\| > \varepsilon\}} \|g_\tau\| d\mu \\ &\leq \mu(\Omega - E) + (1/\varepsilon) \int_E \|g_\tau\| d\mu \\ &\leq \mu(\Omega - E) + (1/\varepsilon)|G|(E), \end{aligned}$$

since $\int_E \|g_\tau\| d\mu \leq |G|(E) < \delta\varepsilon/2 + (1/\varepsilon)\delta\varepsilon/2 < \delta$, by the choice of E and ε . This implies $\lim_\tau g_\tau = 0$ in μ -measure and that $\lim_\tau (h_\tau + g_\tau) = h_0 \in L^1(\mu, \mathfrak{X})$ in μ -measure. This completes the proof of the first statement.

To finish the proof of the theorem, let $T=N$, form the martingales $\{f_n, B_n\}$, $\{g_n, B_n\}$ and $\{h_n, B_n\}$, $f_n = g_n + h_n$ as above. As noted above, $\lim_{n \rightarrow \infty} h_n = h_0$ exists strongly in $L^1(\mu, \mathfrak{X})$. Therefore, a theorem of A. and C. Ionescu-Tulcea and Neveu [15, p. 189], [9, p. 117], $\lim_n h_n(\omega) = h_0(\omega)$ strongly in \mathfrak{X} for μ -almost all $\omega \in \Omega$. On the other hand, $\{\|g_n\|, B_n\}$ is an L^1 -bounded submartingale which converges to zero in measure and therefore almost everywhere. Hence $\lim_n f_n(\omega) = h_0(\omega)$ strongly in \mathfrak{X} for μ -almost all $\omega \in \Omega$. Q.E.D.

It is not immediately obvious that this theorem does, in fact, contain the result of Chatterji [4, Theorem 2] as advertised. To proceed in this direction, a definition is needed. A Banach space \mathfrak{X} has the Radon-Nikodým property with respect to μ if every \mathfrak{X} -valued μ -continuous countably additive set function of bounded variation defined on Σ admits the representation

$$F(E) = \int_E f d\mu, \quad E \in \Sigma$$

for some $f \in L^1(\mu, \mathfrak{X})$. All reflexive spaces and separable dual spaces have the Radon-Nikodým property with respect to an arbitrary finite measure μ .

The next result shows (ii) of the hypothesis of Theorem 4 is redundant in the case that \mathfrak{X} has the Radon-Nikodým property with respect to μ .

THEOREM 5. *Let \mathfrak{X} have the Radon-Nikodým property with respect to μ . Then any martingale $\{f_\tau, B_\tau, \tau \in T\}$ with*

$$\sup_\tau \|f_\tau\|_1 \leq M < \infty$$

has the property that given an $\varepsilon > 0$ there exists a norm (hence weakly) compact convex subset $K \subset \mathfrak{X}$ such that for any choice of $\delta > 0$ there is an index $\tau_0 \in T$ and $E_0 \in B_{\tau_0}$ with $\mu(\Omega - E_0) < \varepsilon$ such that for $\tau \geq \tau_0$

$$\int_E f_\tau d\mu \in \mu(E)K + \delta U$$

for $E \subset E_0$, $E \in B_\tau$.

Proof. As in the proof of Theorem 4, write $\lim_\tau \int_E f_\tau d\mu = F(E) = G(E) + H(E)$, $E \in \bigcup_\tau B_\tau$, where by the current hypothesis, F is of bounded variation and so are G and H . H is μ -continuous and countably additive, while G is singular with respect to μ . Now, as in [19, p. 39] H has a μ -continuous countably additive extension of bounded variation to $\sigma(\bigcup_\tau B_\tau)$. Therefore since \mathfrak{X} has the Radon-Nikodým property with respect to μ there exists $h \in L^1(\mu, \mathfrak{X})$ such that $H(E) = \int_E h d\mu$ for all $E \in \bigcup_\tau B_\tau$. Now let $\varepsilon > 0$ be given. According to Proposition 1, there exists a norm compact convex set K such that for any fixed $\delta > 0$ there is $E' (=E'(\delta, K)) \in \bigcup_\tau B_\tau$ with $\mu(\Omega - E') < \varepsilon/2$ such that

$$H(E) \in \mu(E)K + (\delta/2)\dot{U}.$$

Also since G is μ -singular, there exists a set $E'' \in \bigcup_\tau B_\tau$ with $\mu(\Omega - E'') < \varepsilon/2$ and

$|G|(E'') < \delta/2$. ($|G|$ is the variation of G .) Setting $E_0 = E' \cap E''$, one finds easily that $\mu(\Omega - E_0) < \varepsilon$ and $F(E) \in \mu(E)K + \delta U$ for $E \subset E_0$, $E \in \bigcup_{\tau} B_{\tau}$. But if $\tau_0 \in T$ is chosen such that $E_0 \in B_{\tau_0}$ then

$$\int_E f_{\tau} d\mu = F(E) \in \mu(E)K + \delta U$$

for $\tau \geq \tau_0$ and $E \subset E_0$, $E \in B_{\tau}$. This completes the proof of the theorem.

An immediate consequence of Theorems 4 and 5 is

COROLLARY 6 (CHATTERJI [4, THEOREM 2]). *If \mathfrak{X} satisfies the Radon-Nikodým property with respect to μ then any martingale indexed by N , $\{f_n, B_n\}$ with*

$$\sup_{n \geq 1} \int_{\Omega} \|f_n\| d\mu \leq M < \infty$$

converges almost everywhere.

Theorem 5 can also be used to deduce the theorems of Scalora [17], Chatterji [3] and Ionescu-Tulcea [9] on mean convergence directly from the sufficiency part of Theorem 2.

4. A generalization of the Radon-Nikodým-Bochner theorem. Although the Radon-Nikodým theorem fails when the assumption of countable additivity is weakened to finite additivity, Bochner [1], [6, IV. 9.14] proved the following theorem.

THEOREM (BOCHNER). *Let \mathcal{F} be a field of subsets of a point set Ω . Let μ be a nonnegative finite finitely additive set function defined on \mathcal{F} . If F is a real valued, finitely additive function defined on \mathcal{F} , of bounded variation and μ -continuous, then for each $\varepsilon > 0$, there is a μ -integrable simple function f_{ε} such that the function F_{ε} defined by the equation $F_{\varepsilon}(E) = \int_E f_{\varepsilon} d\mu$, $E \in \mathcal{F}$ satisfies the inequality $|F - F_{\varepsilon}|(\Omega) < \varepsilon$.*

This theorem has been treated at length in the literature [2], [13], and [7]. In [19, Theorem III. 3] the author generalized this theorem to include \mathfrak{X} -valued functions for reflexive Banach spaces \mathfrak{X} . The purpose of this section is to use Proposition 1 to characterize, for general Banach spaces \mathfrak{X} , the \mathfrak{X} -valued finitely additive set functions which allow such an approximation. The integral is that of [6, Chapter III].

THEOREM 6. *Let \mathcal{F} be a field of subsets of a point set Ω . Let μ be a nonnegative finite finitely additive set function defined on \mathcal{F} . Let F be a finitely additive \mathfrak{X} -valued set function defined on \mathcal{F} . Then for each $\varepsilon > 0$ there exists an \mathfrak{X} -valued μ -integrable f_{ε} such that the function F_{ε} defined on \mathcal{F} by the equation $F_{\varepsilon}(E) = \int_E f_{\varepsilon} d\mu$, $E \in \mathcal{F}$, satisfies $|F - F_{\varepsilon}|(\Omega) < \varepsilon$ if and only if*

- (i) F is of bounded variation;
- (ii) F is μ -continuous, and

(iii) given $\varepsilon > 0$ there exists a weakly (norm) compact convex subset $K \subset \mathfrak{X}$ such that for any $\delta > 0$ there exists $E_\delta \in \mathcal{F}$, $\mu(\Omega - E_\delta) < \varepsilon$ such that $F(E) \in \mu(E)K + \delta U$ for $E \subset E_\delta$, $E \in \mathcal{F}$.

Proof (Sufficiency). By [6, IV. 9.10–11] there exists a totally disconnected compact Hausdorff space S_1 such that

(a) Σ_1 , the field of all clopen subsets of S_1 is isomorphic to \mathcal{F} under a Boolean isomorphism $\tau: \mathcal{F} \rightarrow \Sigma_1$; and

(b) every finitely additive real valued set function on Σ_1 is countably additive.

Define F_1 and μ_1 on Σ_1 by $F_1(\tau(E)) = F(E)$ and $\mu_1(\tau(E)) = \mu(E)$, respectively. Now, μ_1 is finite and has a unique extension, still denoted by μ_1 , to $\sigma(\Sigma_1)$, the σ -field generated by Σ_1 . Also since μ_1 is countably additive and F_1 is clearly μ_1 -continuous on Σ_1 , F_1 is countably additive. Now, to prove the sufficiency of the stated conditions let $\varepsilon > 0$ be given, and note that because of the action of the isomorphism $\tau: \mathcal{F} \rightarrow \Sigma_1$, F_1 evidently satisfies the hypothesis of Proposition 1. Hence there exists a μ_1 -measurable function $f_1: S_1 \rightarrow \mathfrak{X}$ such that

$$F_1(E) = \int_E f_1 d\mu_1, \quad E \in \Sigma_1.$$

But simple functions of the form $f = \sum_{i=1}^n x_i \chi_{\tau(E_i)}$, $E_i \in \Sigma_1$ are dense in $L^1(\mu_1, \mathfrak{X})$ by [6, III. 8.3]. Hence there exist $\{E_i\}_{i=1}^n \subset \mathcal{F}$ and $\{x_i\}_{i=1}^n \subset \mathfrak{X}$ such that the μ_1 -simple function $g_1 = \sum_{i=1}^n x_i \chi_{\tau(E_i)}$ satisfies $\int_{S_1} \|f_1 - g_1\| d\mu_1 < \varepsilon$. Now define f_ε on Ω by $f_\varepsilon = \sum_{i=1}^n x_i \chi_{E_i}$ and F_ε on \mathcal{F} by the equation $F_\varepsilon(E) = \int_E f_\varepsilon d\mu$, $E \in \mathcal{F}$. Then clearly

$$|F - F_\varepsilon|(\Omega) \leq |F_1 - G_1|(S_1) \quad (\text{actually there is equality here})$$

$$= \int_{S_1} \|f_1 - g_1\| d\mu_1 < \varepsilon,$$

where $G_1(E) = \int_E g_1 d\mu_1$, $E \in \sigma(\Sigma_1)$. This proves the sufficiency of the stated conditions.

To prove the necessity of the conditions, note that (i) and (ii) are clear. To prove (iii), choose the μ -simple functions

$$f_n = \sum_{i=1}^{m_n} x_{ni} \chi_{E_{ni}}, \quad x_{ni} \in \mathfrak{X}, E_{ni} \in \mathcal{F},$$

$$F_n(E) = \int_E f_n d\mu, \quad E \in \mathcal{F}$$

such that $|F - F_n|(\Omega) < 1/n$. Define \tilde{f}_n on S_1 by

$$\tilde{f}_n = \sum_{i=1}^{m_n} x_{ni} \chi_{\tau(E_{ni})}.$$

Since

$$\begin{aligned} \int_{S_1} \|\tilde{f}_n - \tilde{f}_m\| d\mu_1 &= \int_{\Omega} \|f_n - f_m\| d\mu \\ &\leq |F_n - F|(\Omega) + |F - F_m|(\Omega) < 1/n + 1/m, \end{aligned}$$

it follows that \tilde{f}_n is a Cauchy sequence in $L^1(\mu_1, \mathfrak{X})$ and hence converges to a function $\tilde{f} \in L^1(\mu_1, \mathfrak{X})$. Let $\varepsilon > 0$ be given. By Proposition 1, there exists a weakly (norm) compact convex set $K \subset \mathfrak{X}$ such that given $\delta > 0$ there is $\tau(E_\delta) \in \Sigma_1$ (Σ_1 generates $\sigma(\Sigma_1)$) $\mu_1(S_1 - \tau(E_\delta)) < \varepsilon$ and

$$\int_{\tau(E)} \tilde{f} d\mu_1 \in \mu_1(\tau(E))K + \delta U$$

for $E \subset E_\delta$ (i.e. $\tau(E) \subset \tau(E_\delta)$, $E \in \mathcal{F}$). But

$$\int_E f_n d\mu = \int_{\tau(E)} \tilde{f}_n d\mu_1$$

and

$$F(E) = \lim_n \int_E f_n d\mu = \lim_n \int_{\tau(E)} \tilde{f}_n d\mu_1 = \int_{\tau(E)} \tilde{f} d\mu_1$$

for all $E \in \mathcal{F}$. Hence

$$F(E) = \int_{\tau(E)} \tilde{f} d\mu_1 \in \mu_1(\tau(E))K + \delta U = \mu(E)K + \delta U$$

for $E \subset E_\delta$, $E \in \mathcal{F}$. This proves the necessity of (iii) and the theorem. Q.E.D.

REMARK. Theorem 6 provides an explicit characterization of the $S^1(\mathfrak{X})$ subspaces of the $V^1(\mathfrak{X})$ spaces studied in [19].

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